
Common Fixed Point Theorems for Generalized Contraction Mappings in Partially Ordered Set

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Abstract

In this paper, we prove some common fixed point results in the framework of metric and partially ordered metric spaces satisfying a generalized contractive condition of rational type. The proved results generalize and extend some known results in the literature.

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1. Introduction

The Banach contraction mapping is one of the pivotal results of analysis. It is a very popular tool for solving existence problems in many different fields of mathematics. There are a lot of generalizations of Banach contraction principle in the literature (see [1]-[23] and references cited therein).

Ran and Reurings [22] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and Rodriguez-Lopez [20] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [4] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied results on a first order differential equation with periodic boundary conditions.

Recently, many researchers have obtained fixed point, common fixed point results in metric spaces and partially ordered metric spaces. The purpose of this paper is to establish some common fixed point results satisfying a generalized contraction mappings of rational type in metric and partially ordered metric spaces.

To start with, we recall some definitions.

Definition 1.1. Let M be a nonempty subset of metric space (X, d) and $T, f: M \rightarrow M$. A point x is a common fixed (respectively, coincidence) point of f and T if $x = fx = Tx$ (respectively, $fx = Tx$). The set of fixed points (respectively, coincidence points) of f and T is denoted by $F(f, T)$ (respectively $C(f, T)$).

The pair (T, f) is called

a) commutative if $Tfx = fTx$ for all x ;

a) commutative if $Tfx = fTx$ for all $x \in M$;

b) compatible [16] if $\lim d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $Tx_n = \lim fx_n = t$ for some t in M ;

c) weakly compatible [17] if f and T commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$.

2 Main Results

2.1 Common fixed point theorems in metric spaces

In this section, we prove a common fixed point theorem for three single-valued mappings in the setting of metric spaces.

Theorem 2.1. Let M be a subset of a metric space (X, d) . Suppose that $T, f, g: M \rightarrow M$ satisfy

$$d(Tx, fy) \leq \alpha \left(\frac{d(gx, Tx)d(gy, fy)}{d(gx, gy) + d(gx, fy) + d(gy, Tx)} \right) + \beta(d(gx, gy)) \quad (2.1)$$

for all $x, y \in M$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Suppose also that $T(M) \cup f(M) \subseteq g(M)$ and $(g(M), d)$ is complete. Then

(i) T, f and g have a coincidence point in M ;

(2) If the pairs (g, T) and (g, f) are weakly compatible, then T, f and g have a unique common fixed point.

Proof : Let $x_0 \in X$. Since $T(M) \cup f(M) \subseteq g(M)$ we can choose $x_1, x_2 \in M$ so that $gx_1 = Tx_0$ and $gx_2 = fx_1$. By induction, we construct a sequence $\{x_n\}$ in X such that $gx_{2n+1} = Tx_{2n}$ and $gx_{2n+2} = fx_{2n+1}$, for every $n \geq 0$.

By (2.1),

$$\begin{aligned} d(gx_{2n+2}, gx_{2n+1}) &= d(fx_{2n+1}, Tx_{2n}) \\ &\leq \alpha \left(\frac{d(gx_{2n+1}, fx_{2n+1})d(gx_{2n}, Tx_{2n})}{d(gx_{2n+1}, gx_{2n}) + d(gx_{2n+1}, Tx_{2n}) + d(gx_{2n}, fx_{2n+1})} \right) + \\ &\quad \beta(d(gx_{2n+1}, gx_{2n})) \\ &\leq \alpha \left(\frac{d(gx_{2n+1}, gx_{2n+2})d(gx_{2n}, gx_{2n+1})}{d(gx_{2n+1}, gx_{2n}) + d(gx_{2n+1}, gx_{2n+1}) + d(gx_{2n}, gx_{2n+2})} \right) + \\ &\quad \beta(d(gx_{2n+1}, gx_{2n})) \\ &\leq \alpha \left(\frac{d(gx_{2n+1}, gx_{2n+2})d(gx_{2n}, gx_{2n+1})}{d(gx_{2n+1}, gx_{2n+1})} \right) + \beta(d(gx_{2n+1}, gx_{2n})) \\ &= \alpha(d(gx_{2n+1}, gx_{2n})) + \beta(d(gx_{2n+1}, gx_{2n})) \\ &= (\alpha + \beta)d(gx_{2n+1}, gx_{2n}), \end{aligned} \quad (2.2)$$

Which implies that

$$d(gx_{2n+2}, gx_{2n+1}) \leq (\alpha + \beta)d(gx_{2n+1}, gx_{2n}) \quad (2.3)$$

Using, mathematical induction we have

$$d(gx_{2n+2}, gx_{2n+1}) \leq (\alpha + \beta)^{2n+1} d(gx_{2n+1}, gx_{2n}) \quad (2.4)$$

Put $k = \alpha + \beta < 1$. Now, we shall prove that $\{gx_n\}$ is a Cauchy sequence. For $m \geq n$, we have

$$\begin{aligned} d(gx_m, gx_n) &\leq d(gx_m, gx_{m-1}) + d(gx_{m-1}, gx_{m-2}) + \dots + d(gx_{n+1}, gx_n) \\ &\leq (k^{m-1} + k^{m-2} + \dots + k^n) d(gx_1, gx_0) \\ &\leq \left(\frac{k^n}{1-k} \right) d(gx_1, gx_0) \end{aligned} \quad (2.5)$$

Which implies that $d(gx_m, gx_n) \rightarrow 0$ as $m, n \rightarrow \infty$. Thus $\{gx_n\}$ is a Cauchy sequence.

As $(g(M), d)$ is complete, there is $t \in M$ such that $gx_n \rightarrow gt$ as $n \rightarrow \infty$. We shall prove that t is a coincidence point of T, f and g . We have

$$\begin{aligned} d(gx_{2n+2}, ft) &= d(Tx_{2n}, ft) \\ &\leq \alpha \left(\frac{d(gx_{2n}, Tx_{2n+1}) d(gt, ft)}{d(gx_{2n}, gt) + d(gx_{2n}, ft) + d(gt, gx_{2n+1})} \right) + \beta(d(gx_{2n}, gt)) \\ &= \alpha \left(\frac{d(gx_{2n}, gx_{2n+1}) d(gt, ft)}{d(gx_{2n}, gt) + d(gx_{2n}, ft) + d(gt, gx_{2n+1})} \right) + \beta(d(gx_{2n}, gt)) \end{aligned}$$

On letting $n \rightarrow \infty$, we have $d(gt, ft) = 0$ and hence $gt = ft$.

Also, we have

$$\begin{aligned} d(Tt, gt) &= d(Tt, ft) \\ &\leq \alpha \left(\frac{d(gt, Tt) d(gt, ft)}{d(gt, gt) + d(gt, ft) + d(gt, Tt)} \right) + \beta(d(gt, gt)) \end{aligned}$$

This implies that $d(Tt, gt) = 0$, that is, $Tt = gt$.

Thus we have, $gt = Tt = ft$, that is, t is a coincidence point of T, f and g . Then (i) holds.

Now, suppose that the pairs (g, T) and (g, f) are weakly compatible. Let $z = ft = gt = Tt$. Then we have $gTz = Tgz$ and $gft = fgt$, which implies that $Tz = fz = gz$. On the other hand, we have

$$\begin{aligned} d(gz, z) &= d(Tz, ft) \\ &\leq \alpha \left(\frac{d(gz, Tz) d(gt, ft)}{d(gz, gt) + d(gz, ft) + d(gt, Tz)} \right) + \beta(d(gz, gt)) \end{aligned}$$

This implies that $d(gz, z) = 0$, that is, $gz = z$. Hence, we get that $z = gz = Tz = fz$, that is, z is a common fixed point of g, T and f . This makes end to the proof.

Suppose now that $z^1 \in M$ is another common fixed point of g, T and f , that is

$$z^1 = gz^1 = Tz^1 = fz^1.$$

We have

$$\begin{aligned} d(z, z^1) &= d(Tz, fz^1) \\ &\leq \alpha \left(\frac{d(gz, Tz) d(gz^1, fz^1)}{d(gz, gz^1) + d(gz, fz^1) + d(gz^1, Tz)} \right) + \beta(d(gz, gz^1)). \end{aligned}$$

This implies that $d(z, z^1) = 0$, that is, $z = z^1$. Thus we proved the uniqueness of the common fixed point. Hence, (ii) hold.

Corollary 2.2. Let M be a subset of a metric space (X, d) Suppose that $T, S : M \rightarrow M$ satisfy

$$d(Tx, Ty) \leq \alpha \left(\frac{d(Sx, Tx) d(Sy, Ty)}{d(Sx, Sy) + d(Sx, Ty) + d(Sy, Tx)} \right) + \beta(d(Sx, Sy))$$

for all $x, y \in M$ and for some $\alpha + \beta \in [0, 1)$ with $\alpha + \beta < 1$. Suppose also that $T(M) \subseteq S(M)$ and $(S(M), d)$ is complete. Then

- (i) T and S have a coincidence point in M ;
- (ii) If the pair (S, T) is weakly compatible, then T and S have a unique common fixed point.

Corollary 2.3. [15] Let T be a continuous self map defined on a complete metric space (X, d) . Suppose that T satisfies the following contractive conditions:

$$d(Tx, Ty) \leq \alpha \left(\frac{d(x, Tx) d(y, Ty)}{d(x, y) + d(x, Ty) + d(y, Tx)} \right) + \beta (d(x, y))$$

for all $x, y \in X$, and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$, then T has a unique fixed point in X .

2.2 Common fixed point theorems in ordered metric spaces

In this section, we prove a common fixed point theorem in the setting of ordered metric spaces.

Definitions 2.1. Suppose (X, \leq) is a partially ordered set and $T : X \rightarrow X$. T is said to be monotone nondecreasing if for all $x, y \in X$,

$$x \leq y \text{ implies } Tx \leq Ty. \quad (2.6)$$

Definition 2.2. Let (X, \leq) be a partially ordered set and $T, f, g : X \rightarrow X$ are mappings such that $T(X) \subseteq g(X)$ and $f(X) \subseteq g(X)$. Then T and f are weakly increasing with respect to g if and only if for all $x \in X$, we have

- (a) $Tx \leq fy$ for all $y \in g^{-1}(Tx)$;
- (b) $fx \leq Ty$ for all $y \in g^{-1}(fx)$.

Various examples of such mappings are given in [18].

Remark 2.4. If $gx = x$ for all $x \in X$, then T and f are weakly increasing with respect to g implies that T and f are weakly increasing mappings. Note that the concept of weakly increasing mappings was introduced by Altun and Simsek in [3].

Theorem 2.5. Let (X, \leq) be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $T, f, g : X \rightarrow X$ satisfy

$$d(Tx, fy) \leq \alpha \left(\frac{d(gx, Tx) d(gy, fy)}{d(gx, gy) + d(gx, fy) + d(gy, Tx)} \right) + \beta (d(gx, gy)) \quad (2.7)$$

for all $x, y \in X$, $gx \leq gy$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.

Suppose that

- (i) $T(X) \subseteq g(X)$, $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X ;
- (ii) T and f are weakly increasing with respect to g .

Also suppose that either

- a) the pair (T, g) is compatible and T, g are continuous; or
- b) the pair (f, g) is compatible and f, g are continuous.

Then T, f and g have a coincidence point, that is, there exists $t \in X$ such that $gt = ft = Tt$.

Proof. Let $x_0 \in X$. From (i), we can choose $x_1, x_2 \in X$ such that $gx_1 = Tx_0$ and $gx_2 = fx_1$. By induction, we construct a sequence $\{gx_n\}$ in X such that $gx_{2n+1} = Tx_{2n}$ and $gx_{2n+2} = fx_{2n+1}$, for every $n \geq 0$.

We claim that

$$gx_n \leq gx_{n+1} \text{ for all } n \geq 1. \quad (2.8)$$

Since T and f are weakly increasing mappings with respect to g , we obtain

$$gx_1 = Tx_0 \leq fy, \forall y \in g^{-1}(Tx_0).$$

Since $gx_1 = Tx_0$, then $x_1 \in g^{-1}(Tx_0)$, and we get

$$gx_1 = Tx_0 \leq fx_1 = gx_2$$

Again,

$$gx_2 = fx_1 \preceq Ty, \forall y \in g^{-1}(fx_1).$$

Since, $x_2 \in g^{-1}(fx_1)$ we get

$$gx_2 = fx_1 \preceq Tx_2 = gx_3$$

By induction on n , we conclude that

$$gx_1 \preceq gx_2 \preceq \dots \preceq gx_{2n+1} \preceq gx_{2n+2} \preceq \dots$$

Thus our claim (2.8) holds.

Since $gx_{2n} \leq gx_{2n+1}$ for all $n \geq 1$, from (2.7), we have

$$\begin{aligned} d(gx_{2n+2}, gx_{2n+1}) &= d(fx_{2n+1}, Tx_{2n}) \\ &\leq \alpha \left(\frac{d(gx_{2n+1}, fx_{2n+1}) d(gx_{2n}, Tx_{2n})}{d(gx_{2n+1}, gx_{2n}) + d(gx_{2n+1}, Tx_{2n}) + d(gx_{2n}, fx_{2n+1})} \right) + \\ &\quad \beta(d(gx_{2n+1}, gx_{2n})) \\ &= \alpha \left(\frac{d(gx_{2n+1}, gx_{2n+2}) d(gx_{2n}, gx_{2n+1})}{d(gx_{2n+1}, gx_{2n}) + d(gx_{2n+1}, gx_{2n+1}) + d(gx_{2n}, gx_{2n+2})} \right) + \\ &\quad \beta(d(gx_{2n+1}, gx_{2n})) \\ &\leq \alpha \left(\frac{d(gx_{2n+1}, gx_{2n+2}) d(gx_{2n}, gx_{2n+1})}{d(gx_{2n+1}, gx_{2n+2})} \right) + \beta(d(gx_{2n+1}, gx_{2n})) \\ &= \alpha(d(gx_{2n+1}, gx_{2n})) + \beta(d(gx_{2n+1}, gx_{2n})) \\ &= (\alpha + \beta) d(gx_{2n+1}, gx_{2n}), \end{aligned} \quad (2.9)$$

Which implies that

$$d(gx_{2n+2}, gx_{2n+1}) \leq (\alpha + \beta) d(gx_{2n+1}, gx_{2n}) \quad (2.10)$$

Using, mathematical induction we have

$$d(gx_{2n+2}, gx_{2n+1}) \leq (\alpha + \beta)^{2n+1} d(gx_{2n+1}, gx_{2n}) \quad (2.11)$$

Put $k = \alpha + \beta < 1$. Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. For $m \geq n$, we have

$$\begin{aligned} d(gx_m, gx_n) &\leq d(gx_m, gx_{m-1}) + d(gx_{m-1}, gx_{m-2}) + \dots + d(gx_{n+1}, gx_n) \\ &\leq (k^{m-1} + k^{m-2} + \dots + k^n) d(gx_1, gx_0) \\ &\leq \left(\frac{k^n}{1-k} \right) d(gx_1, gx_0) \end{aligned} \quad (2.12)$$

Which implies that $d(gx_m, gx_n) \rightarrow 0$ as $m, n \rightarrow \infty$. Thus $\{gx_n\}$ is a Cauchy sequence.

Since $(g(X), d)$ is complete, there is $t \in M$ such that $gx_n \rightarrow gt$ as $n \rightarrow \infty$.

Suppose that condition (a) holds. Let $z = gt$. Then we have

$$\lim_{n \rightarrow \infty} Tx_{2n} = \lim_{n \rightarrow \infty} gx_{2n} = z.$$

Since the pair (T, g) is compatible, then

$$\lim_{n \rightarrow \infty} d(g(Tx_{2n}), T(gx_{2n})) = 0. \quad (2.13)$$

Also, from the continuity of T and g , we have

$$\lim_{n \rightarrow \infty} d(g(Tx_{2n}), T(gx_{2n})) = d(gz, Tz). \quad (2.14)$$

Now, using (2.13) and (2.14), by the uniqueness of the limit, we have $d(gz, Tz) = 0$, that is, $gz = Tz$.

Using (2.7), we have

$$d(gz, z) = d(Tz, fz) \\ \leq \alpha \left(\frac{d(gz, Tz) d(gz, fz)}{d(gz, gz) + d(gz, fz) + d(gz, Tz)} \right) + \beta (d(gz, gz)),$$

which implies that $gz = fz$. Thus, we have $gz = fz = Tz$, and z is a coincidence point of T, g and f .
If condition (b) holds, then by following the same arguments, we get the result.

Corollary 2.6. Let (X, \leq) be a partially ordered set and suppose that there exists a metric d and X such that (X, d) is a complete metric space. Suppose that $T, S : X \rightarrow X$ satisfy

$$d(Tx, Ty) \leq \alpha \left(\frac{d(Sx, Tx) d(Sy, Ty)}{d(Sx, Sy) + d(Sx, Ty) + d(Sy, Tx)} \right) + \beta (d(Sx, Sy)) \quad (2.15)$$

for all $x, y \in X$, $Sx \leq Sy$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.

Suppose that

- (i) $T(X) \subseteq S(X)$, and $S(X)$ is a complete subspace of X ;
- (ii) T is monotone S -non-decreasing;
- (iii) the pair (T, S) is compatible and T, S are continuous.

Then T and S have a coincidence point, that is, there exists $t \in X$ such that $St = Tt$.

Other consequences of our results are the following for the mappings involving contractions of integral type.

Denote by Λ the set of functions $\mu : [0, \infty) \rightarrow [0, \infty)$ satisfying the following hypotheses

:

(h1) μ is a Lebesgue-integrable mapping on each compact subset of $[0, \infty)$;

(h2) for any $\varepsilon > 0$, we have $\int_0^\varepsilon \mu(t) dt > 0$.

Corollary 2.7. Let (X, \leq) be a partially ordered set and suppose that there exists a metric d and X such that (X, d) is a complete metric space. Suppose that $T, f, g : X \rightarrow X$ satisfy

$$\int_0^{d(Tx, fy)} \psi(t) dt \leq \alpha \int_0^{\frac{d(gx, Tx) d(gy, fy)}{d(gx, gy) + d(gy, Tx) + d(gx, fy)}} \psi(t) dt + \beta \int_0^{d(g(x), g(y))} \psi(t) dt$$

for all $x, y \in X$ for which $gx \leq gy$ are comparable, $\psi \in \Lambda$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.

Suppose that

- (i) $T(X) \subseteq g(X)$, $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X ;
- (ii) T and f are weakly increasing with respect to g .

Also suppose that either

- (a) the pair (T, g) is compatible and T, g are continuous; or
- (b) the pair (f, g) is compatible and f, g are continuous.

Then T, f and g have a coincidence point, that is, there exists $t \in X$ such that $gt = ft = Tt$.

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